

# DRINFELD CENTER OF ENRICHED MONOIDAL CATEGORIES

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**ABSTRACT.** We show that every modular tensor category can be realized in a canonical way as the Drinfeld center of an enriched monoidal category.

The purpose of this article is to convey a simple observation. A modular tensor category  $\mathcal{B}$  is the Drinfeld center of an enriched monoidal category  $\mathcal{B}^\sharp$ , which is canonically obtained from  $\mathcal{B}$  (see Corollary 2.5). We give a minimal presentation of this result. To justify the definitions introduced here, we include several supporting facts in remarks without giving proofs. A thorough study of this subject and its applications in mathematics and physics are in preparation.

## 1. ENRICHED (MONOIDAL) CATEGORIES

First, we recall the notion of a (monoidal) category enriched in a (braided) monoidal category. See [Ke, MP] and references therein. Let  $\mathcal{B}$  be a monoidal category with tensor unit  $\mathbf{1}$  and tensor product  $\otimes : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$ .

A *category  $\mathcal{C}^\sharp$  enriched in  $\mathcal{B}$*  consists of a set of objects  $Ob(\mathcal{C}^\sharp)$ , an object  $\text{Hom}_{\mathcal{C}^\sharp}(x, y) \in \mathcal{B}$  for every pair  $x, y \in \mathcal{C}^\sharp$ , and a morphism  $\circ : \text{Hom}_{\mathcal{C}^\sharp}(y, z) \otimes \text{Hom}_{\mathcal{C}^\sharp}(x, y) \rightarrow \text{Hom}_{\mathcal{C}^\sharp}(x, z)$  for every triple  $x, y, z \in \mathcal{C}^\sharp$ , such that there exists a morphism  $\text{id}_x : \mathbf{1} \rightarrow \text{Hom}_{\mathcal{C}^\sharp}(x, x)$  for every  $x \in \mathcal{C}^\sharp$  rendering the following diagrams commutative for  $x, y, z, w \in \mathcal{C}^\sharp$ :

$$\begin{array}{ccc}
 & \text{Hom}_{\mathcal{C}^\sharp}(x, y) \otimes \text{Hom}_{\mathcal{C}^\sharp}(x, x) & \\
 \text{Id} \otimes \text{id}_x \nearrow & & \searrow \circ \\
 \text{Hom}_{\mathcal{C}^\sharp}(x, y) & \xrightarrow{\text{Id}} & \text{Hom}_{\mathcal{C}^\sharp}(x, y), \\
 \\
 & \text{Hom}_{\mathcal{C}^\sharp}(y, y) \otimes \text{Hom}_{\mathcal{C}^\sharp}(x, y) & \\
 \text{id}_y \otimes \text{Id} \nearrow & & \searrow \circ \\
 \text{Hom}_{\mathcal{C}^\sharp}(x, y) & \xrightarrow{\text{Id}} & \text{Hom}_{\mathcal{C}^\sharp}(x, y), \\
 \\
 \text{Hom}_{\mathcal{C}^\sharp}(z, w) \otimes \text{Hom}_{\mathcal{C}^\sharp}(y, z) \otimes \text{Hom}_{\mathcal{C}^\sharp}(x, y) & \xrightarrow{\text{Id} \otimes \circ} & \text{Hom}_{\mathcal{C}^\sharp}(z, w) \otimes \text{Hom}_{\mathcal{C}^\sharp}(x, z) \\
 \circ \otimes \text{Id} \downarrow & & \downarrow \circ \\
 \text{Hom}_{\mathcal{C}^\sharp}(y, w) \otimes \text{Hom}_{\mathcal{C}^\sharp}(x, y) & \xrightarrow{\circ} & \text{Hom}_{\mathcal{C}^\sharp}(x, w).
 \end{array}$$

The *underlying category* of  $\mathcal{C}^\sharp$  is a category  $\mathcal{C}$  which has the same objects as  $\mathcal{C}^\sharp$  and  $\text{Hom}_{\mathcal{C}}(x, y) = \text{Hom}_{\mathcal{B}}(\mathbf{1}, \text{Hom}_{\mathcal{C}^\sharp}(x, y))$ .

An *enriched functor*  $F : \mathcal{C}^\sharp \rightarrow \mathcal{D}^\sharp$  between enriched categories in  $\mathcal{B}$  consists of a map  $F : Ob(\mathcal{C}^\sharp) \rightarrow Ob(\mathcal{D}^\sharp)$  and a morphism  $F : \text{Hom}_{\mathcal{C}^\sharp}(x, y) \rightarrow \text{Hom}_{\mathcal{D}^\sharp}(F(x), F(y))$

for every pair  $x, y \in \mathcal{C}^\sharp$  such that the following diagrams commute for  $x, y, z \in \mathcal{C}^\sharp$ :

$$\begin{array}{ccc}
& \mathbf{1} & \\
\text{id}_x \swarrow & & \searrow \text{id}_{F(x)} \\
\text{Hom}_{\mathcal{C}^\sharp}(x, x) & \xrightarrow{F} & \text{Hom}_{\mathcal{D}^\sharp}(F(x), F(x)), \\
\text{Hom}_{\mathcal{C}^\sharp}(y, z) \otimes \text{Hom}_{\mathcal{C}^\sharp}(x, y) & \xrightarrow{\circ} & \text{Hom}_{\mathcal{C}^\sharp}(x, z) \\
F \otimes F \downarrow & & \downarrow F \\
\text{Hom}_{\mathcal{D}^\sharp}(F(y), F(z)) \otimes \text{Hom}_{\mathcal{D}^\sharp}(F(x), F(y)) & \xrightarrow{\circ} & \text{Hom}_{\mathcal{D}^\sharp}(F(x), F(z)).
\end{array}$$

An enriched functor  $F : \mathcal{C}^\sharp \rightarrow \mathcal{D}^\sharp$  induces a functor between the underlying categories  $F : \mathcal{C} \rightarrow \mathcal{D}$ .

An *enriched natural transformation*  $\xi : F \rightarrow G$  between two enriched functors  $F, G : \mathcal{C}^\sharp \rightarrow \mathcal{D}^\sharp$  is a natural transformation between the underlying functors such that the following diagram commutes for  $x, y \in \mathcal{C}^\sharp$ :

$$\begin{array}{ccc}
\text{Hom}_{\mathcal{C}^\sharp}(x, y) & \xrightarrow{G} & \text{Hom}_{\mathcal{D}^\sharp}(G(x), G(y)) \\
F \downarrow & & \downarrow - \circ \xi_x \\
\text{Hom}_{\mathcal{D}^\sharp}(F(x), F(y)) & \xrightarrow{\xi_y \circ -} & \text{Hom}_{\mathcal{D}^\sharp}(F(x), G(y)).
\end{array}$$

**Example 1.1.** Suppose  $\mathcal{B}$  is rigid. Then  $\mathcal{B}$  is canonically promoted to a self-enriched category  $\mathcal{B}^\sharp$ :  $Ob(\mathcal{B}^\sharp) = Ob(\mathcal{B})$ ,  $\text{Hom}_{\mathcal{B}^\sharp}(x, y) = y \otimes x^*$  where  $x^*$  is the left dual of  $x$ , the composition  $\circ : (z \otimes y^*) \otimes (y \otimes x^*) \rightarrow z \otimes x^*$  is induced by the counit map  $v_y : y^* \otimes y \rightarrow \mathbf{1}$ , and  $\text{id}_x$  is given by the unit map  $u_x : \mathbf{1} \rightarrow x \otimes x^*$ .

Now we assume  $\mathcal{B}$  is braided monoidal category with braiding  $c_{x,y} : x \otimes y \rightarrow y \otimes x$ .

Let  $\mathcal{C}^\sharp, \mathcal{D}^\sharp$  be categories enriched in  $\mathcal{B}$ . The *Cartesian product*  $\mathcal{C}^\sharp \times \mathcal{D}^\sharp$  is a category enriched in  $\mathcal{B}$  defined as follows:

- $Ob(\mathcal{C}^\sharp \times \mathcal{D}^\sharp) = Ob(\mathcal{C}^\sharp) \times Ob(\mathcal{D}^\sharp)$ ;
  - $\text{Hom}_{\mathcal{C}^\sharp \times \mathcal{D}^\sharp}((x, y), (x', y')) = \text{Hom}_{\mathcal{C}^\sharp}(x, x') \otimes \text{Hom}_{\mathcal{D}^\sharp}(y, y')$ ;
  - the composition
- $$\begin{aligned}
\circ : \text{Hom}_{\mathcal{C}^\sharp \times \mathcal{D}^\sharp}((x', y'), (x'', y'')) \otimes \text{Hom}_{\mathcal{C}^\sharp \times \mathcal{D}^\sharp}((x, y), (x', y')) \\
\rightarrow \text{Hom}_{\mathcal{C}^\sharp \times \mathcal{D}^\sharp}((x, y), (x'', y''))
\end{aligned}$$

is given by

$$\begin{aligned}
& \text{Hom}_{\mathcal{C}^\sharp}(x', x'') \otimes \text{Hom}_{\mathcal{D}^\sharp}(y', y'') \otimes \text{Hom}_{\mathcal{C}^\sharp}(x, x') \otimes \text{Hom}_{\mathcal{D}^\sharp}(y, y') \\
& \xrightarrow{\text{Id} \otimes c^{-1} \otimes \text{Id}} \text{Hom}_{\mathcal{C}^\sharp}(x', x'') \otimes \text{Hom}_{\mathcal{C}^\sharp}(x, x') \otimes \text{Hom}_{\mathcal{D}^\sharp}(y', y'') \otimes \text{Hom}_{\mathcal{D}^\sharp}(y, y') \\
& \xrightarrow{\circ \otimes \circ} \text{Hom}_{\mathcal{C}^\sharp}(x, x'') \otimes \text{Hom}_{\mathcal{D}^\sharp}(y, y'').
\end{aligned}$$

**Remark 1.2.** We have a canonical equivalence  $(\mathcal{C}^\sharp \times \mathcal{D}^\sharp) \times \mathcal{E}^\sharp \simeq \mathcal{C}^\sharp \times (\mathcal{D}^\sharp \times \mathcal{E}^\sharp)$  for enriched categories  $\mathcal{C}^\sharp, \mathcal{D}^\sharp, \mathcal{E}^\sharp$ . However,  $\mathcal{C}^\sharp \times \mathcal{D}^\sharp \not\simeq \mathcal{D}^\sharp \times \mathcal{C}^\sharp$  in general unless  $\mathcal{B}$  is symmetric. So, the categories enriched in  $\mathcal{B}$  together with the enriched functors and enriched isomorphisms form a monoidal (2,1)-category  $\text{Cat}^\mathcal{B}$ . Then we have the notions of associative algebras, modules and duality in  $\text{Cat}^\mathcal{B}$  [Lu], but there is no obvious notion of commutativity. Unwinding the definition, we see that an associative algebra in  $\text{Cat}^\mathcal{B}$  is nothing but an enriched monoidal categories

defined below. (Similarly, one defines enriched monoidal functors, enriched module categories, tensor product of enriched module categories, etc.)

**Definition 1.3.** A *monoidal category enriched in  $\mathcal{B}$*  consists of a category  $\mathcal{C}^\sharp$  enriched in  $\mathcal{B}$ , an object  $\mathbf{1}_{\mathcal{C}^\sharp} \in \mathcal{C}^\sharp$ , an enriched functor  $\otimes : \mathcal{C}^\sharp \times \mathcal{C}^\sharp \rightarrow \mathcal{C}^\sharp$ , and enriched isomorphisms  $\lambda : \mathbf{1}_{\mathcal{C}^\sharp} \otimes - \rightarrow \text{Id}_{\mathcal{C}^\sharp}$ ,  $\rho : - \otimes \mathbf{1}_{\mathcal{C}^\sharp} \rightarrow \text{Id}_{\mathcal{C}^\sharp}$ ,  $\alpha : \otimes \circ (\otimes \times \text{Id}_{\mathcal{C}^\sharp}) \rightarrow \otimes \circ (\text{Id}_{\mathcal{C}^\sharp} \times \otimes)$  such that the underlying category  $\mathcal{C}$ , the object  $\mathbf{1}_{\mathcal{C}^\sharp}$ , the underlying functor  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  and the isomorphisms  $\lambda, \rho, \alpha$  define an ordinary monoidal category.

**Remark 1.4.** An enriched monoidal category is *strict* if  $\lambda, \rho, \alpha$  are identities [MP, Definition 2.1]. Actually, every enriched monoidal category is monoidally equivalent to a strict one: one might simply use the MacLane strictness theorem to strictify the underlying monoidal category and then lift to a strict enriched monoidal category. We wonder if there is an even more general notion of a non-strict monoidal category enriched in a braided monoidal category.

**Example 1.5.** Suppose  $\mathcal{B}$  is rigid. Then  $\mathcal{B}$  can be canonically promoted to a monoidal category  $\mathcal{B}^\sharp$  enriched in  $\mathcal{B}$  [MP, Section 2.3]. In fact, one needs to promote  $\otimes : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$  to a well-defined enriched functor. It turns out that one should take  $\otimes : \text{Hom}_{\mathcal{B}^\sharp \times \mathcal{B}^\sharp}((x, y), (x', y')) \rightarrow \text{Hom}_{\mathcal{B}^\sharp}(x \otimes y, x' \otimes y')$  to be  $\text{Id}_{x'} \otimes c_{x^*, y' \otimes y^*} : (x' \otimes x^*) \otimes (y' \otimes y^*) \rightarrow (x' \otimes y') \otimes (x \otimes y)^*$ .

One way to define an enriched braided monoidal category is as follows.

**Definition 1.6.** We say that an enriched functor  $F : \mathcal{C}^\sharp \times \mathcal{D}^\sharp \rightarrow \mathcal{E}^\sharp$  is *commutative* if the assignment  $F^{\text{rev}}(y, x) = F(x, y)$  and the composite morphisms

$$\begin{aligned} F^{\text{rev}} : \text{Hom}_{\mathcal{D}^\sharp \times \mathcal{C}^\sharp}((y, x), (y', x')) &= \text{Hom}_{\mathcal{D}^\sharp}(y, y') \otimes \text{Hom}_{\mathcal{C}^\sharp}(x, x') \\ &\xrightarrow{c} \text{Hom}_{\mathcal{C}^\sharp}(x, x') \otimes \text{Hom}_{\mathcal{D}^\sharp}(y, y') \xrightarrow{F} \text{Hom}_{\mathcal{E}^\sharp}(F^{\text{rev}}(y, x), F^{\text{rev}}(y', x')) \end{aligned}$$

define an enriched functor  $F^{\text{rev}} : \mathcal{D}^\sharp \times \mathcal{C}^\sharp \rightarrow \mathcal{E}^\sharp$ .

An *enriched braided monoidal category* consists of an enriched monoidal category  $\mathcal{C}^\sharp$  with a commutative tensor product  $\otimes$ , and an enriched isomorphism  $\beta : \otimes \rightarrow \otimes^{\text{rev}}$  that defines a braiding for the underlying monoidal category  $\mathcal{C}$ .

**Remark 1.7.** A diagrammatic argument shows that an enriched functor  $F : \mathcal{C}^\sharp \times \mathcal{D}^\sharp \rightarrow \mathcal{E}^\sharp$  is commutative if and only if the following diagram commutes for  $x, x' \in \mathcal{C}^\sharp$ ,  $y, y' \in \mathcal{D}^\sharp$ :

$$\begin{array}{ccc} & \text{Hom}_{\mathcal{C}^\sharp}(x, x') \otimes \text{Hom}_{\mathcal{D}^\sharp}(y, y') & \\ \nearrow c^2 & & \searrow F \\ \text{Hom}_{\mathcal{C}^\sharp}(x, x') \otimes \text{Hom}_{\mathcal{D}^\sharp}(y, y') & \xrightarrow{F} & \text{Hom}_{\mathcal{E}^\sharp}(F(x, y), F(x', y')). \end{array}$$

It follows that  $(F^{\text{rev}})^{\text{rev}} = F$  if  $F$  is commutative.

## 2. DRINFELD CENTER

In what follows, we assume that  $\mathcal{B}$  satisfies the following condition:

- (\*)  $\mathcal{B}$  admits equalizers and the intersection of arbitrary many subobjects of a given object  $x \in \mathcal{B}$  exists.

The Drinfeld center [Ma, JS] of a monoidal category has a straightforward generalization:

**Definition 2.1.** Let  $\mathcal{C}^\sharp$  be a monoidal category enriched in  $\mathcal{B}$ . A *half-braiding* for an object  $x \in \mathcal{C}^\sharp$  is an enriched isomorphism  $b_x : x \otimes - \rightarrow - \otimes x$  between enriched endo-functors of  $\mathcal{C}^\sharp$  such that it defines a half-braiding in the underlying monoidal category  $\mathcal{C}$ .

The *Drinfeld center* of  $\mathcal{C}^\sharp$  is a category  $Z(\mathcal{C}^\sharp)$  enriched in  $\mathcal{B}$  defined as follows:

- an object is a pair  $(x, b_x)$ , where  $x \in \mathcal{C}^\sharp$  and  $b_x$  is a half-braiding for  $x$ ;
- $\text{Hom}_{Z(\mathcal{C}^\sharp)}((x, b_x), (y, b_y))$  is the intersection of the equalizers of the diagrams  $\text{Hom}_{\mathcal{C}^\sharp}(x, y) \rightrightarrows \text{Hom}_{\mathcal{C}^\sharp}(x \otimes z, z \otimes y)$  depicted below for all  $z \in \mathcal{C}^\sharp$

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}^\sharp}(x, y) & \xrightarrow{\otimes \circ (\text{id}_z \otimes \text{Id})} & \text{Hom}_{\mathcal{C}^\sharp}(z \otimes x, z \otimes y) \\ \otimes \circ (\text{Id} \otimes \text{id}_z) \downarrow & & \downarrow - \circ b_{x,z} \\ \text{Hom}_{\mathcal{C}^\sharp}(x \otimes z, y \otimes z) & \xrightarrow{b_{y,z} \circ -} & \text{Hom}_{\mathcal{C}^\sharp}(x \otimes z, z \otimes y); \end{array}$$

- the composition  $\circ$  is induced from that of  $\mathcal{C}^\sharp$ .

**Remark 2.2.** The Drinfeld center  $Z(\mathcal{C}^\sharp)$  has an obvious enriched monoidal structure induced from the enriched monoidal structure of  $\mathcal{C}^\sharp$  and the monoidal structure of the ordinary Drinfeld center  $Z(\mathcal{C})$ . The underlying category of  $Z(\mathcal{C}^\sharp)$  is a full subcategory of  $Z(\mathcal{C})$ .

One can show that  $Z(\mathcal{C}^\sharp)$  is the universal enriched monoidal category  $\mathcal{D}^\sharp$  with the following property:  $\mathcal{D}^\sharp$  is equipped with an enriched monoidal functor  $\mathcal{D}^\sharp \rightarrow \mathcal{C}^\sharp$  such that the composite enriched functors  $L : \mathcal{D}^\sharp \times \mathcal{C}^\sharp \rightarrow \mathcal{C}^\sharp \times \mathcal{C}^\sharp \xrightarrow{\otimes} \mathcal{C}^\sharp$  and  $R : \mathcal{C}^\sharp \times \mathcal{D}^\sharp \rightarrow \mathcal{C}^\sharp \times \mathcal{C}^\sharp \xrightarrow{\otimes} \mathcal{C}^\sharp$  are commutative, and is equipped with an enriched isomorphism  $L \simeq R^{\text{rev}}$  which lifts the underlying monoidal functor  $\mathcal{D} \rightarrow \mathcal{C}$  to a monoidal functor  $\mathcal{D} \rightarrow Z(\mathcal{C})$ . As a consequence,  $Z(\mathcal{C}^\sharp)$  is an enriched braided monoidal category in the sense of Definition 1.6.

**Remark 2.3.** It is possible to define the Drinfeld center  $Z(\mathcal{C}^\sharp)$  alternatively as the enriched category of  $\mathcal{C}^\sharp$ - $\mathcal{C}^\sharp$ -bimodule functors  $\mathcal{C}^\sharp \rightarrow \mathcal{C}^\sharp$  as in the unenriched case.

**Theorem 2.4.** Let  $\mathcal{B}$  be a rigid monoidal category satisfying Condition (\*). Then  $Z(\mathcal{B}^\sharp) \simeq \mathcal{B}'$  where  $\mathcal{B}'$  has the same objects as  $\mathcal{B}^\sharp$  but  $\text{Hom}_{\mathcal{B}'}(x, y)$  is the maximal transparent subobject of  $\text{Hom}_{\mathcal{B}^\sharp}(x, y)$ .

*Proof.* Let  $b_x$  be a half-braiding for an object  $x \in \mathcal{B}^\sharp$ . Since  $b_x$  is an enriched isomorphism, we have a commutative diagram for  $y, z \in \mathcal{B}^\sharp$ :

$$\begin{array}{ccc} \text{Hom}_{\mathcal{B}^\sharp}(y, z) & \xrightarrow{\otimes \circ (\text{Id} \otimes \text{id}_x)} & \text{Hom}_{\mathcal{B}^\sharp}(y \otimes x, z \otimes x) \\ \otimes \circ (\text{id}_x \otimes \text{Id}) \downarrow & & \downarrow - \circ b_{x,y} \\ \text{Hom}_{\mathcal{B}^\sharp}(x \otimes y, x \otimes z) & \xrightarrow{b_{x,z} \circ -} & \text{Hom}_{\mathcal{B}^\sharp}(x \otimes y, z \otimes x). \end{array}$$

Taking  $y = \mathbf{1}$ , we obtain a commutative diagram:

$$\begin{array}{ccc} z & \xrightarrow{\text{Id}_z \otimes u_x} & z \otimes x \otimes x^* \\ (\text{Id}_x \otimes c_{x^*, z}) \circ (u_x \otimes \text{Id}_z) \downarrow & & \downarrow \text{Id} \\ x \otimes z \otimes x^* & \xrightarrow{b_{x,z} \otimes \text{Id}_{x^*}} & z \otimes x \otimes x^*, \end{array}$$

which means  $b_{x,z} = c_{x,z}$ . Conversely,  $c_{x,-} : x \otimes - \rightarrow - \otimes x$  defines an enriched isomorphism, i.e. the following diagram commutes for  $y, z \in \mathcal{B}^\sharp$ :

$$\begin{array}{ccc} z \otimes y^* & \xrightarrow{(\text{Id}_z \otimes c_{y^*, x \otimes x^*}) \circ (\text{Id}_{z \otimes y^*} \otimes u_x)} & (z \otimes x) \otimes (y \otimes x)^* \\ \downarrow (\text{Id}_x \otimes c_{x^*, z \otimes y^*}) \circ (u_x \otimes \text{Id}_{z \otimes y^*}) & & \downarrow \text{Id}_{z \otimes x} \otimes c_{x, y}^* \\ (x \otimes z) \otimes (x \otimes y)^* & \xrightarrow{c_{x, z} \otimes \text{Id}_{x \otimes y}^*} & (z \otimes x) \otimes (x \otimes y)^*. \end{array}$$

Therefore, the assignment  $x \mapsto (x, c_{x,-})$  gives rise to a bijection  $\mathcal{B}' \rightarrow Z(\mathcal{B}^\sharp)$  on objects.

By definition,  $\text{Hom}_{Z(\mathcal{B}^\sharp)}((x, b_x), (y, b_y))$  is the maximal subobject  $\iota : t \hookrightarrow y \otimes x^*$  rendering the following diagram commutative for all  $z \in \mathcal{B}^\sharp$ :

$$\begin{array}{ccccc} t & \xrightarrow{u_z \otimes \iota} & (z \otimes z^*) \otimes (y \otimes x^*) & \xrightarrow{\text{Id}_z \otimes c_{z^*, y \otimes x^*}} & (z \otimes y) \otimes (z \otimes x)^* \\ \downarrow \iota \otimes u_z & & & & \downarrow \text{Id}_{z \otimes y} \otimes b_{x, z}^* \\ (y \otimes x^*) \otimes (z \otimes z^*) & \xrightarrow{\text{Id}_y \otimes c_{x^*, z \otimes z^*}} & (y \otimes z) \otimes (x \otimes z)^* & \xrightarrow{b_{y, z} \otimes \text{Id}_{x \otimes z}^*} & (z \otimes y) \otimes (x \otimes z)^*. \end{array}$$

The commutativity of the diagram amounts to say that the double braiding of  $t$  and  $z^*$  is trivial. It follows that  $t$  is the maximal transparent subobject of  $y \otimes x^*$ .  $\square$

A modular tensor category  $\mathcal{C}$  is automatically enriched in the symmetric monoidal category  $\mathcal{V}$  of finite-dimensional vector spaces, and  $\mathcal{V}$  is canonically embedded in  $\mathcal{C}$ . In this way,  $\mathcal{C}$  is enriched in itself.

**Corollary 2.5.** *Let  $\mathcal{B}$  be a modular tensor category. Then  $Z(\mathcal{B}^\sharp) \simeq \mathcal{B}$ .*

**Remark 2.6.** In recent works [He1, He2], certain unitary modular tensor categories (completed by separable Hilbert spaces) were shown to be the Drinfeld centers of certain categories of solitons, and the latter were proposed as candidates for values of Chern-Simons theory on a point. We expect that a self-enriched modular tensor category  $\mathcal{B}^\sharp$  could be realized as the value of a fully extended Reshetikhin-Turaev TQFT on a point such that the value on a circle is  $\mathcal{B}$ .

### 3. A GENERALIZATION

In this section, we give a generalization of Corollary 2.5 (see Corollary 3.3), which is important for applications in physics. This generalization is inspired by [MP].

Let  $\mathcal{B}$  be a braided monoidal category satisfying Condition (\*). We use  $\bar{\mathcal{B}}$  to denote the same monoidal category  $\mathcal{B}$  but equipped with the anti-braiding  $\bar{c}_{x, y} := c_{y, x}^{-1}$ .

Let  $\mathcal{C}$  be a monoidal category equipped with a braided oplax monoidal functor  $\psi : \bar{\mathcal{B}} \rightarrow Z(\mathcal{C})$ . Then every object  $w \in \mathcal{B}$  is equipped with a half braiding  $b_{\phi(w), -}$  in  $\mathcal{C}$ , where  $\phi : \bar{\mathcal{B}} \rightarrow Z(\mathcal{C}) \rightarrow \mathcal{C}$  is the composition of  $\psi$  with the forgetful functor. Suppose  $\mathcal{C}$  has internal hom in  $\mathcal{B}$ . That is, the functor  $\phi(-) \otimes x : \mathcal{B} \rightarrow \mathcal{C}$  has a right adjoint  $[x, -] : \mathcal{C} \rightarrow \mathcal{B}$  for every  $x \in \mathcal{C}$ . In particular, we have a unit map  $u_w : w \mapsto [x, \phi(w) \otimes x]$  for  $w \in \mathcal{B}$  and a counit map  $v_y : \phi[x, y] \otimes x \rightarrow y$  for  $y \in \mathcal{C}$  associated to the adjunction.

**Construction 3.1.** The monoidal category  $\mathcal{C}$  is canonically promoted to a monoidal category  $\mathcal{C}^\sharp$  enriched in  $\mathcal{B}$ . It has the same objects as  $\mathcal{C}$  and  $\text{Hom}_{\mathcal{C}^\sharp}(x, y) = [x, y]$ . The composition  $\circ : [y, z] \otimes [x, y] \rightarrow [x, z]$  is induced by

$$\phi([y, z] \otimes [x, y]) \otimes x \rightarrow \phi[y, z] \otimes \phi[x, y] \otimes x \xrightarrow{\text{Id}_{\phi[x, y]} \otimes v_y} \phi[y, z] \otimes y \xrightarrow{v_z} z$$

and  $\text{id}_x : \mathbf{1} \rightarrow [x, x]$  is induced by  $\phi(\mathbf{1}) \otimes x \simeq x$ . To promote the tensor product  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  to a well-defined enriched functor, one should take

$$\otimes : \text{Hom}_{\mathcal{C}^\sharp \times \mathcal{C}^\sharp}((x, y), (x', y')) \rightarrow \text{Hom}_{\mathcal{C}^\sharp}(x \otimes y, x' \otimes y')$$

to be the morphism  $[x, x'] \otimes [y, y'] \rightarrow [x \otimes y, x' \otimes y']$  induced by

$$\begin{aligned} \phi([x, x'] \otimes [y, y']) \otimes x \otimes y &\rightarrow \phi[x, x'] \otimes \phi[y, y'] \otimes x \otimes y \\ &\xrightarrow{\text{Id}_{\phi[x, x']} \otimes b_{\phi[y, y'], x} \otimes \text{Id}_y} \phi[x, x'] \otimes x \otimes \phi[y, y'] \otimes y \xrightarrow{v_{x'} \otimes v_{y'}} x' \otimes y'. \end{aligned}$$

**Theorem 3.2.** Let  $\mathcal{C}^\sharp$  be constructed above. The underlying category of the Drinfeld center  $Z(\mathcal{C}^\sharp)$  is the centralizer of  $\bar{\mathcal{B}}$  in  $Z(\mathcal{C})$ . Moreover,  $\text{Hom}_{Z(\mathcal{C}^\sharp)}(x, y)$  for  $x, y \in Z(\mathcal{C}^\sharp)$  represents the functor  $\text{Hom}_{Z(\mathcal{C})}(\psi(-) \otimes x, y) : \mathcal{B}^{\text{op}} \rightarrow \text{Set}$ .

*Proof.* Let  $b_x$  be a half-braiding for an object  $x \in \mathcal{C}^\sharp$ . Since  $b_x$  is an enriched isomorphism, we have a commutative diagram for  $y, z \in \mathcal{C}^\sharp$ :

$$\begin{array}{ccc} [y, z] & \xrightarrow{\otimes \circ (\text{Id} \otimes \text{id}_x)} & [y \otimes x, z \otimes x] \\ \otimes \circ (\text{id}_x \otimes \text{Id}) \downarrow & & \downarrow - \circ b_{x, y} \\ [x \otimes y, x \otimes z] & \xrightarrow{b_{x, z} \circ -} & [x \otimes y, z \otimes x]. \end{array}$$

The diagram is equivalent to the following one via the adjunction between  $\phi(-) \otimes x \otimes y$  and  $[x \otimes y, -]$

$$\begin{array}{ccc} \phi[y, z] \otimes x \otimes y & \xrightarrow{\text{Id}_{\phi[y, z]} \otimes b_{x, y}} & \phi[y, z] \otimes y \otimes x \\ b_{\phi[y, z], x} \otimes \text{Id}_y \downarrow & & \downarrow v_z \otimes \text{Id}_x \\ x \otimes \phi[y, z] \otimes y & \xrightarrow{\text{Id}_x \otimes v_z} x \otimes z \xrightarrow{b_{x, z}} & z \otimes x. \end{array} \quad (3.1)$$

Taking  $y = \mathbf{1}$ , we obtain a commutative diagram:

$$\begin{array}{ccccc} & & \phi[\mathbf{1}, z] \otimes x & & \\ & b_{\phi[\mathbf{1}, z], x} \swarrow & & \searrow v_z \otimes \text{Id}_x & \\ x \otimes \phi[\mathbf{1}, z] & \xrightarrow{b_{x, \phi[\mathbf{1}, z]}} & \phi[\mathbf{1}, z] \otimes x & \xrightarrow{v_z \otimes \text{Id}_x} & z \otimes x. \end{array} \quad (3.2)$$

Let  $z = \phi(w)$  where  $w \in \bar{\mathcal{B}}$ . Note that the functor  $[\mathbf{1}, -] : \mathcal{C} \rightarrow \mathcal{B}$  is right adjoint to  $\phi$ . So, the composition  $z \xrightarrow{\phi(u_w)} \phi[\mathbf{1}, z] \xrightarrow{v_z} z$  is the identity morphism. Therefore, the commutative diagram (3.2) implies that the double braiding of  $(z, b_z)$  and  $(x, b_x)$  is trivial. This shows that  $(x, b_x)$  lies in the centralizer of  $\bar{\mathcal{B}}$  in  $Z(\mathcal{C})$ .

Conversely, if  $(x, b_x)$  lies in the centralizer of  $\bar{\mathcal{B}}$  in  $Z(\mathcal{C})$ , then the diagram (3.1) is clearly commutative for all  $y, z \in \mathcal{C}^\sharp$  thus  $(x, b_x) \in Z(\mathcal{C}^\sharp)$ . This proves the first claim of the theorem.

By definition,  $\text{Hom}_{Z(\mathcal{C}^\sharp)}((x, b_x), (y, b_y))$  is the maximal subobject  $\iota : t \hookrightarrow [x, y]$  rendering the following diagram commutative for all  $z \in \mathcal{C}^\sharp$ :

$$\begin{array}{ccc} t & \xrightarrow{\otimes \circ (\text{id}_z \otimes \iota)} & [z \otimes x, z \otimes y] \\ \otimes \circ (\iota \otimes \text{id}_z) \downarrow & & \downarrow - \circ b_{x,z} \\ [x \otimes z, y \otimes z] & \xrightarrow{b_{y,z} \circ -} & [x \otimes z, z \otimes y]. \end{array}$$

This diagram is equivalent to the following one via the adjunction between  $\phi(-) \otimes x \otimes z$  and  $[x \otimes z, -]$

$$\begin{array}{ccccc} \phi(t) \otimes x \otimes z & \xrightarrow{\phi(\iota) \otimes \text{Id}_x \otimes z} & \phi[x, y] \otimes x \otimes z & \xrightarrow{\text{Id}_{\phi[x, y]} \otimes b_{x,z}} & \phi[x, y] \otimes z \otimes x & \xrightarrow{b_{\phi[x, y], z} \otimes \text{Id}_x} & z \otimes \phi[x, y] \otimes x \\ \downarrow \phi(\iota) \otimes \text{Id}_x \otimes z & & & & & & \downarrow \text{Id}_z \otimes v_y \\ \phi[x, y] \otimes x \otimes z & \xrightarrow{v_y \otimes \text{Id}_z} & y \otimes z & \xrightarrow{b_{y,z}} & z \otimes y. \end{array}$$

Therefore,  $t$  is the maximal subobject such that the composite morphism  $\phi(t) \otimes x \xrightarrow{\phi(\iota) \otimes \text{Id}_x} \phi[x, y] \otimes x \xrightarrow{v_y} y$  in  $\mathcal{C}$  preserves half-braiding, i.e. defines a morphism in  $Z(\mathcal{C})$ . In another word,  $\text{Hom}_{\mathcal{B}}(-, t) \simeq \text{Hom}_{Z(\mathcal{C})}(\psi(-) \otimes (x, b_x), (y, b_y))$ . This proves the second claim of the theorem.  $\square$

**Corollary 3.3.** *Let  $\mathcal{B}$  be a nondegenerate braided fusion category, let  $\mathcal{C}$  be an indecomposable multi-fusion category equipped with a fully faithful braided monoidal functor  $\bar{\mathcal{B}} \hookrightarrow Z(\mathcal{C})$  and let  $\mathcal{C}^\sharp$  be the enriched monoidal category as in Construction 3.1. Then  $Z(\mathcal{C}^\sharp) \simeq \mathcal{B}'$  where  $\mathcal{B}'$  is the centralizer of  $\bar{\mathcal{B}}$  in  $Z(\mathcal{C})$ .*

*Proof.* Since  $\mathcal{B}, \mathcal{C}$  are semisimple,  $\mathcal{C}$  has internal hom in  $\mathcal{B}$ , thus  $\mathcal{C}^\sharp$  is well defined. We have  $Z(\mathcal{C}) \simeq \bar{\mathcal{B}} \boxtimes \mathcal{B}'$  by [DGNO, Theorem 3.13]. This implies that  $\text{Hom}_{Z(\mathcal{C}^\sharp)}(x, y)$  lies in the abelian subcategory of  $\mathcal{B}$  generated by the tensor unit  $\mathbf{1}$ , which is nothing but the category of finite-dimensional vector spaces. Therefore,  $\text{Hom}_{Z(\mathcal{C}^\sharp)}(x, y) \simeq \text{Hom}_{Z(\mathcal{C})}(x, y)$ , as desired.  $\square$

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## REFERENCES

- [DGNO] V. G. Drinfeld, S. Gelaki, D. Nikshych, V. Ostrik, *On braided fusion categories*, Selecta Mathematica 16 (2010) 1-119 [arXiv:0906.0620].
- [He1] A. Henriques, *What Chern-Simons theory assigns to a point*, arXiv:1503.06254 [math-ph].
- [He2] A. Henriques, *Bicommutant categories from conformal nets*, arXiv:1701.02052 [math.OA].
- [JS] A. Joyal, R. Street, *Tortile Yang-Baxter operators in tensor categories*, J. Pure Appl. Algebra 71 (1991), no. 1, 43-51.
- [Ke] G. M. Kelly, *Basic concepts of enriched category theory*, Repr. Theory Appl. Categ. vol. 10 (2005).
- [Lu] J. Lurie, *Higher algebras*, a book available at: <http://www.math.harvard.edu/~lurie/papers/higheralgebra.pdf>.
- [Ma] S. Majid, *Representations, duals and quantum doubles of monoidal categories*, Rend. Circ. Mat. Palermo (2) Suppl. No. 26 (1991), 197-206.
- [MP] S. Morrison, D. Penneys, *Monoidal categories enriched in braided monoidal categories*, arXiv:1701.00567 [math.CT].

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